DCL: Synthesis and Design Workshop
The Future of Embodied Design for Mathematical Imagination and Cognition

Mitchell J. Nathan, University of Wisconsin-Madison, mnathan@wisc.edu
Caro Williams-Pierce, University of Maryland, carowp@umd.edu
Candace Walkington, Southern Methodist University, cwalkington@smu.edu
Dor Abrahamson, University of California, Berkeley, dor@berkeley.edu
Erin Ottmar, Worcester Polytechnic Institute, erottmar@wpi.edu
Hortensia Soto, University of Northern Colorado, hortensia.soto@unco.edu
Martha W. Alibali, University of Wisconsin-Madison, mwalibali@wisc.edu

Keywords: design, embodiment, cognition, imagination, mathematics

The State of the Art for the Workshop Theme

The next generation of digital learning environments for science, technology, engineering, and mathematics (STEM) content will be more interactive, more personalized, more affordable, and more collaborative. ***Embodied design*** offers an emerging cluster of research-oriented pedagogical frameworks for the principled design of learning environments for promoting mathematics education. These environments will provide developers and teachers with rich, real-time data from users to track engagement, support formative assessment, and target both conscious and unconscious learning processes. Mathematics education -- the connective representational "language" among the STEM fields -- benefits from such innovation as it supports the creative process in STEM for imagining what can be. Yet, such creativity and curiosity, which the Mathematical Association of America (Zorn, 2015) promotes as a vehicle to mathematical proficiency and communication, remains out of reach for many students. This workshop focused on the potential of designing for embodied mathematical imagination and cognition (EMIC), with the goal of specifying principles that broaden the mathematical participation of young learners through body-based forms of reasoning and expression.

Embodied design for mathematics education, rooted in a broad reading of the Learning Sciences and practiced through design-based approaches, addresses many of these futuristic qualities while supporting the improvement of instructional methodology for students' mathematical proficiency in the classroom. Our collective work thus far has identified the promise of EMIC for mathematical learning in various ways. For example, Abrahamson and Bakker (2016) explored how students' physical enactment of the covariation of two constant (but unequal) rates using hand motions detected by Wii and tablet platforms fostered an understanding of proportional reasoning. This goal-directed, sensorimotor, mediated activity allowed unschooled students to give rise to formal and generalizable mathematical ideas (Abrahamson, 2015). Ottmar, Landy, and Goldstone (2012) describe interventions where students physically engage with algebraic expressions through movable tiles or touchpad technologies. By using rigid motion as a perceptual grounding mechanism, students dynamically interact with and transform algebraic symbol systems. Enyedy, Danish, and DeLiema (2015) used mixed-reality to support students' understanding of Newtonian force using motion tracking. They found that verbal and physical reflection on embodied activity and first-person embodied play allowed students to engage deeply with challenging concepts. Nathan and Walkington (2017) showed how directed actions coupled with pedagogical hints presented in a motion capture video game increased players' dynamic gesture production, which improved students' mathematical insights and valid proof for geometry conjectures. Notably, a subsequent study showed that mathematically relevant directed actions during game play benefitted high school students significantly more than "irrelevant" directed actions, provided that students gestured while constructing their mathematical justifications (Walkington et al., under review).

In order to develop broader design principles that subsume these independent efforts, and foster learning across mathematical content and digital platforms, we hosted a workshop at the University of Wisconsin–Madison that brought together leading scholars in mathematical reasoning, teaching, and learning who work on embodied design. We sought to attract an interdisciplinary set of scholars and practitioners from education research, cognitive science, the learning sciences, developmental psychology, dance, movement science, computer science, and mathematics and science education. The research presented spanned K–16 topics in content areas such as arithmetic and algebra, proportional reasoning and fractions, geometry, complex numbers and functions, statistics, and calculus. We focused on the design of systems for classroom settings, with attention to equity and access for underrepresented groups, while examining evidence of learning both in and outside of school. *The overall general goal for our workshop was to form a ten-year research agenda that* *would provide a coherent set of evidence-based design principles for enhancing mathematics education and broadening participation in all STEM fields.* The findings, principles, and proposed future research extend to studies of mathematical intuition and reasoning, learning in and outside of formal educational settings, teacher professional development, classroom instruction, and assessment.

In this white paper, we briefly review the EMIC literature, describe the participants and the structure of the workshop, and report on principles and findings that emerged from the workshop. We conclude with recommendations for near-term, moderate term, and long-term EMIC design and research goals. For more information on the workshop, participants, topics, and schedule, see www.embodiedmathematics.com. When we share direct quotes from attendees, they are from a follow-up survey conducted after the workshop -- this survey is discussed in more detail in later sections.

Literature Review

Reviewing the state of the art of mathematics education, Schoenfeld (2016) notes that "both embodied cognition (e.g., Nemirovsky & Ferrara, 2009) and embodied design (e.g., Abrahamson, 2009; Alibali & Nathan, 2012) are receiving increased attention in mathematics education" (p. 514). A number of existing systems used in classrooms, afterschool settings, and randomized control experiments have been built to integrate kinesthetic motion capture, real-time optical motion tracking, and touch screens that use perceptual-motor simulation of continuous motion to enact symbol manipulation (e.g., Dittman et al., 2017; Ng & Sinclair, 2015; Ottmar & Landy, 2017). Each approach offers causal accounts of how directed movement instruction enables students to develop new sensorimotor schemes that facilitate their mathematical reasoning. Because these schemes are self-generated and contextualized, they provide grounded understandings of the mathematical relations and operations that they embody, which, in turn, facilitate learning new math concepts.

The current EMIC field offers an emerging set of evidence-based findings in support of embodied design principles for promoting mathematics learning through movement (e.g., Abrahamson et al., 2016; Nathan & Walkington, 2017; Ottmar & Landy, 2017; Smith, King, & Hoyte, 2014; Williams-pierce et al., 2017). These studies are redefining data collection and analysis methods, including automated analyses of learners' speech, data analytic methods for body movement, gestures, and eye gaze motion, along with assessments of students' content knowledge and problem solving performance, learning, and mathematical intuition (Flood et al., 2015; Soto-Johnson, 2014; Soto-Johnson & Troup, 2014; Walkington et al., 2018). Evidence is mounting that actions can influence cognition via automatic, unconscious (System 1; Kahneman, 2011) processes, such as mathematical intuitions (Nathan, 2017); whereas instruction that engages language and reflection help connect tacit learning processes to the deliberate, verbal processes that typically mark System 2 (Trninic & Abrahamson, 2012). Understanding how conscious and unconscious processes interact opens up new forms of learning and teaching. At the same time, advancements in data management, multimodal and spatial analysis, analytic methods for text, audio, video, motion capture, eye tracking, user logs, brain imaging, and social media has spawned rapid growth in the opportunities to analyze, classify, and uncover latent patterns in multimodal discourse (e.g., Abrahamson & Bakker, 2016; Alibali et al., 2014; de Freitas & Sinclair, 2014; Hall, Ma, & Nemirovsky, 2015; Levine & Scollon, 2004; Nathan et al., 2017). Smith, King, and Gonzalez (2016) developed an analytic method for multimodal analysis using arm movements and verbal reports to detect differences among elementary grade children learning to classify fractions. Williams-Pierce (2017) combined physical gestures, digital actions by in-game avatars, and spoken dialog to understand children's collaborative exploration in a video microworld. Others (e.g., Nathan et al., 2018; Pier et al., 2014; Walkington et al., in press) have applied automated text data mining methods to spoken transcripts along with gesture data to detect learners' exploratory processes. They found that gesture production accounted for unique predictive power in their models, above and beyond effects for speech, prior knowledge, and attitudes toward math. More broadly, embodied cognition is a rapidly emerging area of importance in philosophy of mind and the social sciences (De Vega et al., 2012; Hall & Nemirovsky, 2012; Lee, 2014; Newen et al., 2018; Shapiro, 2014) and was the conference theme of the 2019 biennial meeting of The International Conference on Computer-Supported Collaborative Learning (CSCL, 2019).

Our workshop occurred at a time of significant convergence of theoretical, technological, and methodological developments in embodied cognition. At the same time, there is a trove of technological, methodological, and theoretical questions that must be addressed before we can formulate and implement effective design principles. Embodied design applies to "tools whose operatory function is engineered specifically so as to … cultivate … the development of particular sensorimotor schemes as a condition for masterful control of the environment in accord with task demands" and thereby "come to ground the mathematical concepts we want these students to learn" (Abrahamson & Bakker, 2016, p. 5). Lindgren and Johnson-Glenberg (2013) offer several "precepts" for the design of embodied and mixed-reality learning environments that emphasize "gestural congruency," where learners' actions are structurally or analogically related to the symbols and their meaning" (p. 446). This illustrates Lee's (2014) observation that as embodied cognition gains prominence in education, so, too, do new ways of using technology to support teaching and learning. These new uses of technology, in turn, offer novel opportunities for students and scientists to engage in math visualization, symbolization, intuition, and reasoning. In order for these designs to successfully scale up, they must be informed by research that demonstrates both ecological and internal validity. Consequently, our workshop focused specifically on developing a coherent set of evidence-based design principles for enhancing mathematics education and broadening participation in STEM fields.

Participants and Process

The organizing committee advertised the workshop to a wide variety of communities, and screened the resulting formal applications. Forty-nine applications were accepted, and nearly all of them were fully funded to attend. Our attendees were primarily from the United States, but we had six international participants. Each applicant was asked to present a poster or paper, or facilitate an educator round table or embodied mathematical activity, and we designed a single-track program based upon their contributions. As one attendee noted[1]: "Because everyone participated in some form, the workshop had a greater sense of collaboration and investment than other experiences I've had"; and as another shared, "Size and mix of talks, activities, breaks was perfect -- I feel I got to meaningfully interact with almost everyone there."

One of our goals with EMIC was to support participation and inclusion of educators and researchers across different developmental levels, and to infuse our workshop with expertise from various adjacent fields. To that end, we conducted outreach to K-20 teachers and instructors. A teacher participant said: "As a teacher, I found this invaluable. I have been pushed to think of education in a whole new light. ... I have made connections with impressive people who look to enhance the face of learning... What was so humbling about this week was the fact that these incredible people wanted … to listen to the challenges I face and celebrate the growth I experience. I cannot thank these organizers and participants enough for helping me emerge as a stronger, more well-rounded educator." We also reached out to graduate students and postdoctoral scholars, and recruited keynote speakers and discussants who are renowned scholars in their own arenas. A survey respondent said that they were particularly pleased at "the opportunities for colleagues at a range of career points to contribute [and] the inclusion of teachers."

We accepted 5 K-20 teachers, each of whom conducted their own educator roundtable to discuss how EMIC informs teaching, and how teaching in turn informs EMIC. We accepted 19 graduate students and postdoctoral/early career faculty, who each presented at the opening day poster session. We accepted 21 senior faculty who each presented for 10 to 15 minutes. Lastly, a mixture of students, postdoctoral scholars, and faculty (the majority of whom also presented a poster or a talk) facilitated one of 9 embodied mathematical activities throughout the workshop to *ground* (Barsalou, 2008; Nathan, 2008) our discussions.

Our two keynote speakers were Dr. Brian Bottge, who specializes in mathematics and special education, and Dr. Maxine McKinney de Royston, who specializes in mathematics and multicultural education related to race, identity, and equity. They were chosen specifically to help push our EMIC conversations towards the importance of equity in mathematical learning. A survey respondent noted that they "particularly liked...the choice of keynotes (providing fresh perspectives on embodied learning)." Our two discussants were Dr. Arthur Glenberg, an expert in language and embodied cognition, and Dr. Jim Slotta, an expert in technology and learning. They infused our workshop with considerations from relevant, adjacent fields of inquiry.

Principles

Three sections in our follow-up survey were designed specifically to identify common statements and areas of literature across the attendees, given the scholarly diversity of our workshop. Represented at the workshop were EMIC experts from the fields of: learning sciences, psychology, dance, movement science, linguistics, computer science, education and special education, and mathematics. These statements and authors were mined from the group notes taken on the third day of the workshop, where we asked attendees to discuss the workshop and the EMIC field more broadly. Here, we share only those principles or authors which 50% or more of respondents identified with.

We asked participants whether or not they agreed with certain statements or typically used literature from certain fields. 71% definitely agreed with the statement: "More experimental evidence is needed to pinpoint the specific interactions between students, subject matter, and situated, grounded, and embodied

[1] All direct quotes in this paper come from a post-workshop survey sent out to all the participants. 33 of the 49 attendees completed the survey. All percentages have been rounded to the nearest whole number.

curricular design.” 68% definitely agreed with the statement: “I operate under the assumption that all cognition is inherently embodied.” 61% definitely agreed with: “I am still learning about cognition, and how we learn.” Participants then noted that they typically used the following frameworks, fields, or approaches in their research and design: gesture theory (75%); the tenet that social emotion can impact conceptual integration (63%); principles on designing dynamic gesture-based interfaces for mathematics (56%); that learners should ‘experience first, signify later’ (a direct quote from an attendee’s presentation that caught the collective interest; 56%); that we should encourage physical and spatial exploration of the structure of algebra (50%); the belief that dynamic gestures allow participants to physically experience generalized properties through enactment (56%); and distributed cognition (50%). Finally, the top four influential authors were Alibali, Nathan, Lakoff & Núñez, and Abrahamson.

Surprises and Tensions

An emergent tension that became a critical theme throughout the workshop was the practical educational implications of EMIC, and how to support the innovations and findings from the field in transitioning into direct impact on our educational system. Attendees agreed that: teachers are professionals that should be deeply involved in developing, adopting, and adapting embodied learning activities (81%); there are unanswered questions regarding working with practitioners that need investigation (75%); and teachers should be able to adopt and adapt body-based learning activities (75%). Given that one of our goals in developing this workshop was to include teachers -- as attendees and contributors -- the emergence of this tension is a wonderful side effect of valuing the voices of practitioners in EMIC, and further supports the field in moving toward a more direct impact on classrooms and learners.

Attendees also agreed with some other surprises and tensions that we identified and then included in the survey: there are unanswered questions regarding different types of embodiment that need investigation (84%); activities should be designed to leverage our naturally occurring perceptions toward conceptual understanding (72%); what counts as math (and who decides) is a constant tension in this interdisciplinary field (63%); and there are unanswered questions regarding research practices that need investigation (59%).

Overall Synthesis and Recommendations

Alongside the Principles and Surprises and Tensions described above, we wish to share five other points of importance to the EMIC community: research; theory; practice; theory-practice relations; and reflection, planning, and resources. First, our EMIC workshop (and EMIC research more broadly) was characterized by lively discussions that reflect the many diverse and complementary methodological considerations of the emerging inter-discipline -- not unsurprising, given the plethora of fields represented, as described above. Second, we found a distinct desire for a theoretical convergence that leverages interdisciplinary connections into a coherent collective voice. In particular, our attendees seek to identify consistent learning principles that inform EMIC while still preserving the distinct disciplinary identities.

Thirdly, there is a need to consider how the EMIC perspective broadens participation for those who are minoritized or differently abled; increases the awareness of gesture as signifying content-related communication; considers the classroom environment, curricular objects, and teacher needs when introducing technology; supports learning by beginning from intuition; and enables movement. Fourthly, there was a distinct sense of urgency for participatory design, and an emphasis on the need to keep including teachers when were are designing experiences for the classroom. In addition, there is a push for evidence that gesture directly facilitates learning and instruction. Lastly, attendees noted that the EMIC collective needs further reflection, planning, and resources, and that we need to strategize for community development, intellectual cohesion, identity, longevity, and impact.

In the following three sections, we give recommendations at three timeframes (immediate, near-term, and longer-term). Each of these time frames represent a component of our overall goal *to form a ten-year research agenda that would provide a coherent set of evidence-based design principles for enhancing mathematics education and broadening participation in all STEM fields.*

Recommendations with Immediate Applicability

Several areas are ripe for immediate applicability to design for STEM learning. Cognitive and attentional processes are quite limited. Ways instructional design provides resources to offload cognitive and attention processing to the body and other external resources offer immediate promise for improving intellectual performance. These include the external memory aids such as paper and pen and mathematical manipulatives, designing for distributed and extended cognition (Clark, 2008; Hutchins, 1995), and the strategic use of gestures and actions during task performance (Kirsh & Maglio, 1994) and instruction (Alibali & Nathan, 2012; Edwards, 2009). There is also a body of evidence that conceptual development from novice learners of all ages benefits from early exposure to concrete learning experiences before demonstrating generalized mastery and transfer--what our participants referred to as "experience first, signify later" (Abrahamson, 2009)--which favors approaches such as concreteness fading and progressive formalization (Fyfe, McNeil, Son, & Goldstone, 2014), and a rejection of *formalisms first* approaches to curriculum design and instruction (Nathan, 2012).

It is also important that EMIC learning environments employ participatory design considerations, specifically including teachers and learners who can offer feedback and advice throughout the design process. Doing so is expected to increase the fidelity of implementation and overall usefulness of the product or experience in both classrooms and informal learning environments. Further, guidelines for implementing and facilitating such environments are also crucial. Developing supporting materials for the learning environments, alongside guidelines for "adopting and adapting" to the local context, are important design considerations that can be overlooked. We anticipate more participatory design and implementation recommendations for producing undergraduate EMIC learning environments in Spring 2020, as we have recently been funded to conduct a second EMIC workshop with that focus (Hortensia Soto, PI).

Recommendations for Near-Term (3-5 Years)

We have four design and development recommendations for the near term. First, *cultivate teacher-researcher-policymaker partnerships*. A crucial goal for EMIC is to have direct influence on how and what students learn in mathematics classrooms, but in order to do so, a strong relationship between these three types of stakeholders is required.

Second, it is important to *promote opportunities to share and curate empirical inquiry into the conditions for designing EMIC for instruction, assessment, and learning environments*. This can be done through traditional means (e.g., presenting at conferences, conducting workshops, writing journal articles), but should also be done by sharing professional development materials online, working with textbook developers to include body-based items and assessment, developing EMIC design courses for practitioners such as teachers and librarians, and conducting local outreach at schools, higher education disciplinary departments, and professional schools, including schools of education, and local and regional school board meetings. It also means forming communities of practice for education that cross traditional boundaries and bring in ideas from scholars in dance, kinesiology, sports science and related fields that have much to contribute to emerging theories of conceptual development in mathematics (e.g., Abrahamson, Sánchez–García & Smyth, 2016; belcastro & Schaffer, 2011).

Third, we need to *develop and refine data collection tools and research methods for studying body based processes, body states, and cognitive states*. The growth of motion capture technology, single- and dual-eye tracking, head-mounted fMRI scanners, AR and VR, and a variety of other tools offers important new ways to document behavior and learning (e.g., Shvarts & Abrahamson, 2019). We also need concomitant methodological frameworks for incorporating these data streams into analytically coherent accounts of learning.

Fourth, and potentially most important, *investigate how EMIC addresses equity in the classroom*. One of the benefits of EMIC is that it can easily provide a non-linguistic on-ramp to mathematics, which means that learners who may have a language barrier can have increased access through body-based interventions as well as ways to express their mathematical reasoning in nonverbal ways. Gestures are known to convey the transitional states of learners and can be used to identify learner readiness (Goldin-Meadow, Alibali & Church, 1993). Educational practitioners and researchers need to attend to the mathematical gestures made by learners, so they can better understand what the learners understand -- or don't -- about the topic under discussion, without the constraints of adhering to propositional forms. Learners who have already expressed a dislike of mathematics, are anxious about their math performance, or who consider themselves "unteachable," may experience new points of access to mathematics presented through movement, concrete experiences, and body-based forms of engagement.

Recommendations for Longer-Term (5+ Years)

The EMIC program produced a number of intriguing findings and hypotheses that need to be refined and replicated, which offer promise for advancing STEM learning and performance.

While cognitive offloading readily informs learning environment design, the hypotheses that body-based movements *cause* changes in conceptual states, or even that the body *constitutes* cognition are far more controversial (Shapiro, 2014). These latter positions offer a radical change from traditional notions of cognition, yet they have some support from empirical and philosophical sources. An example of this is gesture-inhibition interventions, which show mixed results for their impact on performance (Walkington et al., 2019).

The locus of action for some embodied design interventions operates outside of the learner's conscious awareness yet can have demonstrable effects on intellectual performance (e.g., Morgan & Abrahamson, 2016). Understanding how these unconscious processes interact with consciousness and metacognitive processes is important to inform educational practices and curricula. For example, when and how to integrate sensorimotor forms of knowing with disciplinary frames of reference seems vital for articulating from mathematical intuitions and early forms of knowing to generalizable mathematical principles and theories (e.g., Abrahamson & Bakker, 2016; Nathan & Walkington, 2017). Multisensory perceptions can help learners perceive structures and patterns in math that might not be available from symbolic representations (Gerofsky, 2007; Sinclair, 2005). This has implications for assessment of knowledge that is encoded in nonverbal form.

Spatial systems are powerful resources for mathematical reasoning, as evident in the utility of diagrams, symbol systems, simulations, and spatial metaphors. However, our spatial reasoning draws not only on visual sensation but also on haptic and proprioceptive sensations, where there is less research with respect to learning, teaching, and assessment. Notably, professional mathematicians use embodied and tactile resources to construct their own understanding and communicate with other scholars (Henderson & Taimina, 2007). An embodied framework for mathematical reasoning is likely to expand those systems to include touch and movement senses in ways that are greatly expansive and inclusive for describing how we experience the world that we know and our own imagination (Abrahamson et al., 2019).

Finally, we close with two broad recommendations to foster EMIC as a mature field of scholarship. First, *consolidate and integrate so that we have a stable, shared theory*. EMIC benefits from and struggles with its interdisciplinary nature. As the EMIC community enacts the convergence of so many fields, the potential for advancing our understanding of embodied mathematical imagination and cognition is immense, but we must develop a coherent locus at the center of the convergence so that we can productively share across and learn from our often disparate fields.

Second, *promote a long-term commitment for funding research on embodied cognition, instruction, and assessment practices*. In order for EMIC to have a direct and productive impact on instruction, assessment, and our understanding of the mechanisms of learning, we must have a stable source of funding that begins at *exploration* and ends at *scaling up*. In other words, EMIC requires a commitment that our interdisciplinary and cutting-edge investigations will be supported from the initial steps to distributing successful products and lesson plans into the schools of the United States and beyond.

Acknowledgements

Many thanks to: David Landy for his assistance with this project; Michael Swart for his incredible support while running the workshop; and Leah Rosenbaum for acting as scheduler and contact person for the workshop's mathematical activities. This material is based upon work supported by the National Science Foundation under grant #1824662 awarded to the PI (first author) and the senior personnel (remaining authors). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. Educational Studies in Mathematics, 70(1), 27–47.

Abrahamson, D. (2015). The monster in the machine, or why educational technology needs embodied design. In V. R. Lee (Ed.), *Learning technologies and the body: Integration and implementation* (pp. 21–38). New York, NY: Routledge.

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. *Cognitive Research: Principles and Implications, 1*(1), 33.

Abrahamson, D., Flood, V. J., Miele, J. A., & Siu, Y.-T. (2019). Enactivism and ethnomethodological conversation analysis as tools for expanding Universal Design for Learning: The case of visually impaired mathematics students. *ZDM Mathematics Education, 51*(2), 291-303. doi:10.1007/s11858-018-0998-1

Abrahamson, D., Sánchez–García, R., & Smyth, C. (2016). Metaphors are projected constraints on action: An ecological dynamics view on learning across the disciplines. Singapore: International Society of the Learning Sciences.

Abrahamson, D., Shayan, S., Bakker, A., & Van der Schaaf, M. F. (2016). Eye-tracking Piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Development, 58*(4-5), 218-244.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286.

Alibali, M. W., Nathan, M. J., Wolfgram, M. S., Church, R. B., Jacobs, S. A., Johnson Martinez, C., & Knuth, E. J. (2014). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. *Cognition and instruction, 32*(1), 65-100.

belcastro, S. M., & Schaffer, K. (2011). Dancing mathematics and the mathematics of dance. Math Horizons, 18(3), 16-20.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom.* New York, NY: Cambridge University Press.

Clark, A. (2008). Supersizing the mind: Embodiment, action, and cognitive extension. OUP USA.

Dittman, M., Soto-Johnson, H., Dickinson, S., & Harr, T. (2017). Game building with complex-valued functions. *PRIMUS, 27*(8),869–879.

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. Educational Studies in Mathematics, 70(2), 127-141.

Enyedy, N., Danish, J. A., & DeLiema, D. (2015). Constructing liminal blends in a collaborative augmented-reality learning environment. *International Journal of Computer-Supported Collaborative Learning, 10*(1), 7–34.

Flood, V. J., Neff, M., & Abrahamson, D. (2015). Boundary interactions: Resolving interdisciplinary collaboration challenges using digitized embodied performances. In O. Lindwall, P. Häkkinen, T. Koschmann, P. Tchounikine, & S. Ludvigsen (Eds.), *Exploring the material conditions of learning: Opportunities and challenges for Computer Supported Collaborative Learning (CSCL) conference,* vol.1 (pp. 94–100).Gothenburg, Sweden: The International Society of the Learning Sciences.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. Educational psychology review, 26(1), 9-25.

Gerofsky, S. (2007). "Because You Can Make Things With It": A Rationale for a Project to Teach Mathematics as a Multimodal Design Tool in Secondary Education. *Journal of Teaching and Learning, 5*(1).

Hall, R., Ma, J. Y., & Nemirovsky, R. (2015). Re-scaling bodies in/as representational instruments in GPS drawing. In V. R. Lee (Ed.), *Learning technologies and the body: Integration and implementation in formal and informal learning environments* (pp. 112–131). New York, NY: Routledge.

Henderson, D. W. & Taimina, D. (2007). Experiencing meanings in geometry. In Sinclair, N., Pimm, D., & Higginson, W. (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity*, (pp. 58-83). New York: Springer-Verlag.

Hutchins, E. (1995). Cognition in the Wild (No. 1995). MIT press.

Kirsh, D., & Maglio, P. (1994). On distinguishing epistemic from pragmatic action. Cognitive science, 18(4), 513-549.

Lee, V. R. (2015). *Learning technologies and the body: Integration and implementation in formal and informal learning environments.* New York, NY: Routledge.

Levine, P., & Scollon, R. (Eds.). (2004). *Discourse and technology: Multimodal discourse analysis.* Washington, D.C.: Georgetown University Press.

Lindgren, R., & Johnson-Glenberg, M. (2013). Emboldened by embodiment: Six precepts for research on embodied learning and mixed reality. *Educational Researcher, 42*(8), 445–452.

Morgan, P., & Abrahamson, D. (2016). Cultivating the ineffable: The role of contemplative practice in enactivist learning. *For the Learning of Mathematics, 36*(3), 31-37.

Nathan, M. J. (2008). An embodied cognition perspective on symbols, gesture, and grounding instruction. *Symbols and embodiment: Debates on meaning and cognition, 18*, 375-396.

Nathan, M. J. (2012). Rethinking formalisms in formal education. Educational Psychologist, 47(2), 125-148.

Nathan, M. J. (2017). One function of gesture is to make new ideas: Evidence for reciprocity between action and cognition. In R. B. Church, M. W. Alibali, & S. D. Kelly, (Eds.) *Why gesture? How the hands function in speaking, thinking and communicating.* (Chapter 8, pp. 175–196). Philadelphia, PA: John Benjamins Publishing Company. doi: 10.1075/gs.7.04der

Nathan, M. J., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. *Cognitive Research: Principles and Implications, 2*(1), 9.

Nathan, M. J., Wolfgram, M., Srisurichan, R., Walkington, C., & Alibali, M. W. (2017). Threading mathematics through symbols, sketches, software, silicon and wood: Teachers produce and maintain cohesion to support STEM integration. *The Journal of Educational Research, 110*(3), 272-293. doi: 10.1080/00220671.2017.1287046

Nathan, M. J., Walkington, C., Vinsonhaler, R., Michaelis, J., McGinty, J., Binzak, J. V., & Kwon, O. H. (2018). *Embodied account of geometry proof, insight, and intuition among novices, experts, and English language learners.* Paper presented to the 2018 Annual Meeting of the American Educational Research Association, New York, NY.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159–174.

Ng, O., & Sinclair, N. (2015). 'Area without numbers': Using Touchscreen dynamic geometry to reason about shape. *Canadian Journal of Science, Mathematics and Technology Education, 15*(1), 84–101.

Ottmar, E., & Landy, D. (2017). Concreteness fading of algebraic instruction: Effects on learning. *Journal of the Learning Sciences, 26*(1), 51–78.

Ottmar, E., Landy, D., & Goldstone, R. (2012, January). Teaching the perceptual structure of algebraic expressions: Preliminary findings from the pushing symbols intervention. *In Proceedings of the Cognitive Science Society* (Vol. 34, No. 34).

Paas, F., & Sweller, J. (2012). An evolutionary upgrade of cognitive load theory: Using the human motor system and collaboration to support the learning of complex cognitive tasks. Educational Psychology Review, 24(1), 27-45.

Pier, E. L., Walkington, C., Williams, C., Boncoddo, R., Waala, J., Alibali, M. W., et al. (2014). Hear what they say and watch what they do: Predicting valid mathematical proofs using speech and gesture. In W. Penuel, S. A. Jurow, & K. O'Connor (Eds.), *Learning and becoming in practice: Proceedings of the Eleventh International Conference of the Learning Sciences* (pp. 649–656). Boulder, CO: University of Colorado.

Schoenfeld, A. H. (2016). Research in mathematics education. *Review of Research in Education, 40*(1), 497–528.

Shvarts, A., & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. Learning, Culture and Social Interaction, 22, 100316.

Sinclair, N. (2005). Chorus, colour, and contrariness in school mathematics. THEN: Journal, 12.

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. Gesture, 14(3), 351-374.

Smith, C., King, B. & Gonzalez, D. (2016). Using multimodal learning analytics to identify patterns of interactions in a body-based mathematics activity. *Journal of Interactive Learning Research, 27*(4), 355-379.

Smith, C. P., King, B., & Hoyte, J. (2014). Learning angles through movement: Critical actions for developing understanding in an embodied activity. *The Journal of Mathematical Behavior, 36*, 95–108.

Soto-Johnson, (2014). Visualizing the arithmetic of complex numbers. *International Journal of Technology in Mathematics Education, 21*(3), 103–114.

Soto-Johnson, H., & Troup, J. (2014). Reasoning on the complex plane via inscriptions and gesture. *The Journal of Mathematical Behavior, 36*, 109–125.

Trninic, D., & Abrahamson, D. (2012). Embodied artifacts and conceptual performances. In Proceedings of the international conference of the learning sciences: Future of learning (ICLS 2012)(Vol. 1, pp. 283–290).

Walkington, C., Nathan, M.J., & Wang, M. (under review). The Effect of Relevant Directed Arm Motions on Gesture Usage and Proving of Geometry Conjectures. Submitted to AERA 2020.

Walkington, C., Woods, D., Chelule, G., & Nathan, M. J. (2018). *Does restricting hand gesture impair mathematical reasoning?* Paper to the 2018 Annual Meeting of the American Educational Research Association. New York, NY.

Walkington, C., Woods, D., Nathan, M. J., Chelule, G., & Wang, M. (2019). Does restricting hand gestures impair mathematical reasoning? *Learning and Instruction, 64*, 101225.

Williams-Pierce, C. (2017). Fractions, mental operations, and a unique digital context. Brief Research Report. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 1349–1352). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
Williams-Pierce, C. Pier, E. L., Walkington, C., Boncoddo, R., Clinton, V., Alibali, M. W., & Nathan, M. J. (2017). What we say and how we do: Action, gesture, and language in proving. *Journal for Research in Mathematics Education*, *48*(3), 248-260.

Wilson, M. (2002). Six views of embodied cognition. Psychonomic bulletin & review, 9(4), 625-636.

Zorn, P. (2015). *CUPM guide to majors in the mathematical sciences.* Washington, D.C.: Mathematical Association of America.